



An infinitesimal Noether–Lefschetz theorem for Chow groups

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ABSTRACT

Let X be a smooth, complex projective variety, and Y be a very general, sufficiently ample hypersurface in X . A conjecture of M.V. Nori states that the natural restriction map $\mathrm{CH}^p(X)_{\mathbb{Q}} \rightarrow \mathrm{CH}^p(Y)_{\mathbb{Q}}$ is an isomorphism for all $p < \dim Y$ and an injection for $p = \dim Y$. This is the *generalized Noether–Lefschetz conjecture*. We prove an infinitesimal version of this conjecture.

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1. Introduction

A fundamental theorem concerning the topology of algebraic varieties is the *Weak Lefschetz theorem* (also known as the *Lefschetz hyperplane section theorem*).

Theorem 1. *Let X be a smooth, projective variety of dimension $m + 1$ over the field of complex numbers, and $Y \subset X$ be a hyperplane section. The restriction map of singular cohomology groups $H^i(X, \mathbb{Z}) \rightarrow H^i(Y, \mathbb{Z})$ is an isomorphism for $i < m$, and a monomorphism for $i = m$. Equivalently, one has that $H^i(X, Y; \mathbb{Z}) = 0$ for $i \leq m$.*

The philosophy of motives, and the conjectures of Bloch and Beilinson imply (see e.g. §2, [13] for details) that motivic analogs of the above theorem should also be true, namely that

Conjecture 1 (Weak Lefschetz conjecture). $\mathrm{CH}^p(X) \otimes \mathbb{Q} \rightarrow \mathrm{CH}^p(Y) \otimes \mathbb{Q}$ is an isomorphism for $p < m/2$, and a monomorphism for $p = m/2$.

In the special case when $X = \mathbb{P}^{m+1}$, this conjecture is an old question of Hartshorne (see [7]). Very little is known about this conjecture, except in the case $p = 1$, where the statement even holds integrally. In this case, using the correspondence between divisors and line bundles, the theorem is usually stated as follows:

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Theorem 2 (*Grothendieck–Lefschetz theorem*, [6]). Let X be a smooth, projective variety of dimension at least 4, and Y be a smooth hyperplane section. The restriction map of Picard groups $\text{Pic}(X) \rightarrow \text{Pic}(Y)$ is an isomorphism.

When X is a 3-fold, a slightly weaker result is true.

Theorem 3 (*Noether–Lefschetz theorem*, [2]). Let X be a smooth, projective 3-fold, and Y be a very general, sufficiently ample hypersurface in X . The restriction map $\text{Pic}(X) \rightarrow \text{Pic}(Y)$ is an isomorphism.

Pioneering work in the context of algebraic cycles, especially various refinements and extensions of the Noether–Lefschetz theorem, was carried out by M. Green and C. Voisin, among others, beginning in the 1980's (see [16] for a detailed account of these and related developments). Their results, especially [5] (which was also independently proved by C. Voisin (unpublished)), in turn inspired M. Nori to prove the following remarkable connectivity theorem (see Lemmas 2.1 and 2.2, and Theorem 4, [10]).

Theorem 4. Let X be a smooth, projective variety of dimension $m+1$, and $\mathcal{O}_X(1)$ be a sufficiently ample line bundle. Let $S := |\mathcal{O}_X(1)|$, $A := X \times S$, and $B := \{(x, f) \in X \times S \mid f(x) = 0\}$ be the universal hypersurface. Then for any smooth morphism $g : T \rightarrow S$, one has $H^p(A_T, \Omega_{(A_T, B_T)}^q) = 0$ for $p \leq m$ and $p + q \leq 2m$. Consequently, $H^i(A_T, B_T; \mathbb{Q}) = 0$ for $i \leq 2m$.

Here $\Omega_{(A_T, B_T)}^q$ is defined by the exact sequence

$$0 \rightarrow \Omega_{(A_T, B_T)}^q \rightarrow \Omega_{A_T}^q \rightarrow i_* \Omega_{B_T}^q \rightarrow 0,$$

where $i : B_T \rightarrow A_T$ is the natural inclusion. Let $k = k(S)$ denote the function field of the parameter space S above, and \bar{k} denote its algebraic closure. Let $X_{\bar{k}} := X \times_{\mathbb{C}} \bar{k}$, and $Y := B \times_S \bar{k}$. We have the following consequence of Theorem 4.

Theorem 5. $H^p(X_{\bar{k}}, \Omega_{(X_{\bar{k}}, Y)}^q) = 0$, for $p \leq m$ and $p + q \leq 2m$.

Proof. The result follows from the fact that cohomology and Kähler differentials commute with direct limits. First, note that we can write \bar{k} as the inverse limit of schemes T_α where each $T_\alpha \rightarrow S$ is finite étale over an affine open in S . Note that each T_α is affine (since it is finite over an open affine), and therefore the transition maps in the inverse system A_{T_α} are all affine. It follows that both $\lim_{\leftarrow} A_{T_\alpha}$ and $\lim_{\leftarrow} B_{T_\alpha}$ exist in the category of schemes. Since fiber products commute with taking inverse limits, one has $X_{\bar{k}} \cong \lim_{\leftarrow} A_{T_\alpha}$ and $Y \cong \lim_{\leftarrow} B_{T_\alpha}$. Moreover, the universal property of Kähler differentials implies that $\Omega_{X_{\bar{k}}}^1 \cong \lim_{\rightarrow} \Omega_{A_{T_\alpha}}^1$ and $\Omega_Y^1 \cong \lim_{\rightarrow} \Omega_{B_{T_\alpha}}^1$. Since taking exterior powers commutes with direct limits, we have an analogous result for the higher order Kähler differentials. One also has an analogous statement for the relative differentials, since taking direct limits is an exact functor. Combining everything we have:

$$H^p(X_{\bar{k}}, \Omega_{(X_{\bar{k}}, Y)}^q) \cong H^p(\lim_{\leftarrow} A_{T_\alpha}, \lim_{\rightarrow} \Omega_{(A_{T_\alpha}, B_{T_\alpha})}^q) \cong \lim_{\rightarrow} H^p(A_{T_\alpha}, \Omega_{(A_{T_\alpha}, B_{T_\alpha})}^q).$$

Here the last isomorphism follows from the fact that cohomology commutes with direct limits. The result now follows from Theorem 4. \square

Using his connectivity theorem, Nori proved the existence of non-trivial cycles in the Griffiths group which are in fact not detected by the Abel–Jacobi map, thus generalizing the original result due to Griffiths.

Furthermore, in keeping with the philosophy of motives, he conjectured the following generalization of the Noether–Lefschetz theorem:

Conjecture 2 (see [10], Conjecture 7.2.5). *With notation as above, $\mathrm{CH}^p(X_{\bar{k}}) \otimes \mathbb{Q} \rightarrow \mathrm{CH}^p(Y) \otimes \mathbb{Q}$ is an isomorphism for $p \leq m - 1$, and a monomorphism for $p = m$.*

We note that for $p = 1$, this conjecture is also true integrally and is [Theorem 3](#) above. The reader may refer to [15] to see the equivalence between the two statements. As explained in §3 [13], one can factor the above restriction map as follows: Let $I \cong \mathcal{O}_X(-1)$ be the sheaf of ideals of Y in $X_{\bar{k}}$, and let Y_n be the subscheme with sheaf of ideals I^{n+1} . Let $\mathfrak{X}_{\bar{k}}$ be the completion of $X_{\bar{k}}$ along Y . Then the restriction map in [Conjecture 2](#) factors as

$$\mathrm{H}^p(X_{\bar{k}}, \mathcal{K}_{p, X_{\bar{k}}}) \otimes \mathbb{Q} \rightarrow \mathrm{H}_{\mathrm{cont}}^p(\mathfrak{X}_{\bar{k}}, (\mathcal{K}_{p, Y_n})) \otimes \mathbb{Q} \rightarrow \mathrm{H}^p(Y, \mathcal{K}_{p, Y}) \otimes \mathbb{Q}.$$

Here the middle term is the continuous cohomology of Y , in the sense of Jannsen (see [8]), with values in the pro-sheaf (\mathcal{K}_{p, Y_n}) . One may view the first arrow in the factorization above as the “algebraization” map, and the second one as the “deformation” map. Define $\mathrm{CH}_{\mathrm{cont}}^p(\mathfrak{X}_{\bar{k}}) := \mathrm{H}_{\mathrm{cont}}^p(\mathfrak{X}_{\bar{k}}, (\mathcal{K}_{p, Y_n}))$. In [14] (see Corollary 4.2), we proved the following result.

Theorem 6 (*Infinitesimal weak Lefschetz theorem*). *Let X be a smooth, projective variety of dimension $m + 1$, and $Y \subset X$ be a smooth hyperplane section. The natural restriction map $\mathrm{CH}_{\mathrm{cont}}^p(\mathfrak{X}) \rightarrow \mathrm{CH}^p(Y)$ is an isomorphism for $p < m/2$, and an injection¹ for $p = m/2$.*

In this article, we prove the following infinitesimal version of Nori’s conjecture.

Theorem 7 (*Infinitesimal Noether–Lefschetz theorem*). *Let X be a smooth, projective variety and $\mathcal{O}_X(1)$ be a sufficiently ample line bundle. For $Y \subset X_{\bar{k}}$ as above, $\mathrm{H}_{\mathrm{cont}}^p(\mathfrak{X}_{\bar{k}}, (\mathcal{K}_{q, Y_n})) \rightarrow \mathrm{H}^p(Y, \mathcal{K}_{q, Y})$ is an isomorphism for $p < m$ and $p+q < 2m$, and an injection for $p = m$ and $p+q \leq 2m$. In particular, $\mathrm{CH}_{\mathrm{cont}}^p(\mathfrak{X}_{\bar{k}}) \rightarrow \mathrm{CH}^p(Y)$ is an isomorphism for $p < m$, and a monomorphism for $p = m$.*

Remark 1. We note that both the *infinitesimal Lefschetz theorems* hold integrally, though the conjectures are for Chow groups with rational coefficients.

Remark 2. When $p = 1$, then the above *Infinitesimal Noether–Lefschetz theorem* implies the *Noether–Lefschetz theorem* ([Theorem 3](#) above). Note that by Proposition 3.1, [14], we have $\mathrm{Pic}(\mathfrak{X}_{\bar{k}}) \cong \mathrm{CH}_{\mathrm{cont}}^1(\mathfrak{X}_{\bar{k}})$, and so by Grothendieck’s algebraization theorem (see [6]) for vector bundles, we have $\mathrm{Pic}(X_{\bar{k}}) \cong \mathrm{Pic}(\mathfrak{X}_{\bar{k}}) \cong \mathrm{Pic}(Y)$. The global Noether–Lefschetz theorem now follows by a standard “spreading out” argument (see for instance, §3, [15]).

Remark 3. Effective versions of Nori’s connectivity theorem have been proved in [9,12], and in [12] it has been conjectured that the same should hold even in the motivic version, and consequently for [Theorem 7](#). It follows quite easily from our proof that this is indeed the case.

We end this section with a quick note about the proof: the idea of the proof is exactly as in [13,14] – whereas in those papers, we reduced the proof to Lefschetz connectivity and Kodaira–Nakano vanishings, here the same role is played by Nori’s connectivity theorem and Serre vanishing.

¹ The injectivity part was not stated in [14], but it is immediate from Theorem 4.1 in [14] which is the analog of [Theorem 7](#) above.

2. Cohomological connectivity for infinitesimal thickenings of hypersurfaces

We begin by proving Nori's connectivity theorem for the thickenings Y_n , for $n \gg 0$.

Proposition 1. *With notation as above, we have for $n \gg 0$,*

$$H^p(X_{\bar{k}}, \Omega_{(X_{\bar{k}}, Y_n)}^q) = 0 \text{ for all } p \leq m.$$

Proof. In the following, let $i_n : Y_n \rightarrow X_{\bar{k}}$ denote the natural embedding, and let $i_0 = i$. Consider the following diagram with exact rows:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \Omega_{X_{\bar{k}}}^q(-n-1) & = & \Omega_{X_{\bar{k}}}^q(-n-1) & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \Omega_{(X_{\bar{k}}, Y_n)}^q & \rightarrow & \Omega_{X_{\bar{k}}}^q & \rightarrow & i_{n*}\Omega_{Y_n}^q \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & i_*\Omega_Y^{q-1}(-n-1) & \rightarrow & i_{n*}\Omega_{X_{\bar{k}}|Y_n}^q & \rightarrow & i_{n*}\Omega_{Y_n}^q \rightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \quad (1)$$

The exactness of the bottom row follows from Lemma 3.4 in [14] and exactness of the push-forward map i_{n*} . The middle vertical can be seen to be exact by an application of the projection formula: $i_{n*}\Omega_{X_{\bar{k}}|Y_n}^q \cong \Omega_{X_{\bar{k}}|Y_n}^q$. The left-most vertical is exact by the snake lemma. The long exact sequence in the cohomology associated to the leftmost column yields an exact sequence

$$H^p(\Omega_{X_{\bar{k}}}^q(-n-1)) \rightarrow H^p(\Omega_{(X_{\bar{k}}, Y_n)}^q) \rightarrow H^p(\Omega_Y^{q-1}(-n-1)) \rightarrow H^{p+1}(\Omega_{X_{\bar{k}}}^q(-n-1)). \quad (2)$$

By Serre vanishing (and duality), the first, third and fourth terms vanish for $n \gg 0$ and $p < m$. Therefore, we see that $H^p(\Omega_{(X_{\bar{k}}, Y_n)}^q) = 0$ for $n \gg 0$ if $p < m$. When $p = m$, the first term vanishes for large n , and so we are left with the exact sequence

$$0 \rightarrow H^m(\Omega_{(X_{\bar{k}}, Y_n)}^q) \rightarrow H^m(\Omega_Y^{q-1}(-n-1)) \rightarrow H^{m+1}(\Omega_{X_{\bar{k}}}^q(-n-1)). \quad (3)$$

Let α_q denote the right hand map in the above exact sequence.

Claim. *The map $\alpha_q : H^m(\Omega_Y^{q-1}(-n-1)) \rightarrow H^{m+1}(\Omega_{X_{\bar{k}}}^q(-n-1))$ is injective for $n \gg 0$.*

To prove the claim, we will first note that this map factors as

$$H^m(\Omega_Y^{q-1}(-n-1)) \rightarrow H^m(\Omega_{X_{\bar{k}}|Y}^q(-n)) \rightarrow H^{m+1}(\Omega_{X_{\bar{k}}}^q(-n-1)), \quad (4)$$

and then show that each of these maps is an injection.

To see the factorization, we consider the following diagram, where the top two rows are the exact sequences in the leftmost columns in diagram (1) for $\Omega_{(X_{\bar{k}}, Y_n)}^q$ and $\Omega_{(X_{\bar{k}}, Y_{n-1})}^q$ respectively:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Omega_{X_{\bar{k}}}^q(-n-1) & \rightarrow & \Omega_{(X_{\bar{k}}, Y_n)}^q & \rightarrow & i_* \Omega_Y^{q-1}(-n-1) \rightarrow 0 \\
& & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Omega_{X_{\bar{k}}}^q(-n) & \rightarrow & \Omega_{(X_{\bar{k}}, Y_{n-1})}^q & \rightarrow & i_* \Omega_Y^{q-1}(-n) \rightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & \Omega_{X_{\bar{k}}|Y}^q(-n) & \rightarrow & i_{n*} \Omega_{(Y_n, Y_{n-1})}^q & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array} \tag{5}$$

The left vertical is the usual restriction (to Y) sequence of the bundle $\Omega_{X_{\bar{k}}}^q$, twisted by $\mathcal{O}_{X_{\bar{k}}}(-n)$, and hence is exact. The middle vertical sequence is obtained from the commutative diagram whose two rows are the defining sequences for $\Omega_{(X_{\bar{k}}, Y_n)}^q$ and $\Omega_{(X_{\bar{k}}, Y_{n-1})}^q$, with the obvious maps between them. Finally, note that the right most top vertical is the zero map. A consideration of the associated diagram in cohomology and an application of the snake lemma gives the desired factorization. On the other hand, we have the following commutative diagram with exact rows and column:

$$\begin{array}{ccccccc}
& & 0 & & & & \\
& & \downarrow & & & & \\
0 & \rightarrow & \Omega_{X_{\bar{k}}}^q(-n-1) & \rightarrow & \Omega_{(X_{\bar{k}}, Y_n)}^q & \rightarrow & \Omega_Y^{q-1}(-n-1) \rightarrow 0 \\
& & \downarrow & & & & \\
& & \Omega_{X_{\bar{k}}}^q(-n) & & & & \\
& & \downarrow & & & & \\
0 & \rightarrow & \Omega_{X_{\bar{k}}|Y}^{q-1}(-n-1) & \rightarrow & \Omega_{X_{\bar{k}}|Y}^q(-n) & \rightarrow & \Omega_Y^q(-n) \rightarrow 0 \\
& & \downarrow & & & & \\
& & 0 & & & &
\end{array} \tag{6}$$

As noted above, the exactness of the bottom row is from Lemma 3.4 in [14]. Now the first map in equation (4) is the map between the cohomologies of the first two terms in the bottom row in the above diagram, and the second map in equation (4) is the boundary map of cohomologies of the vertical short exact sequence.

Now consider the cohomology sequence for the bottom row:

$$H^{m-1}(\Omega_Y^q(-n)) \rightarrow H^m(\Omega_Y^{q-1}(-n-1)) \rightarrow H^m(\Omega_{X_{\bar{k}}|Y}^q(-n)); \tag{7}$$

The first term vanishes for $n \gg 0$, and so the map $H^m(\Omega_Y^{q-1}(-n-1)) \rightarrow H^m(\Omega_{X_{\bar{k}}|Y}^q(-n))$ is an injection.

Next, consider the cohomology long exact sequence of the vertical short exact sequence in diagram (6):

$$H^m(\Omega_{X_{\bar{k}}}^q(-n)) \rightarrow H^m(\Omega_{X_{\bar{k}}|Y}^q(-n)) \rightarrow H^{m+1}(\Omega_{X_{\bar{k}}}^q(-n-1)). \tag{8}$$

Once again, we see that the first term vanishes for $n \gg 0$, and so the map $H^m(\Omega_{X_{\bar{k}}|Y}^q(-n)) \rightarrow H^{m+1}(\Omega_{X_{\bar{k}}}^q(-n-1))$ is an injection. \square

Remark 4. We note here that the above result has a straight-forward generalization to the case when $Y \subset X_{\bar{k}}$ above is a complete intersection. In this case, one notes that the term in the top row in (1) is replaced by $\Omega_{X_{\bar{k}}}^q \otimes I^{n+1}$ (where I is the ideal sheaf of Y), and the first term in the bottom row gets replaced by $\Omega(q-1, n) := \ker(i_{n*} \Omega_{X_{\bar{k}}|Y_n}^q \rightarrow i_{n*} \Omega_{Y_n}^q)$. The rest of the argument is exactly as in the proof above; the vanishings $H^p(\Omega_{X_{\bar{k}}}^q \otimes I^{n+1}) = 0$ and $H^p(\Omega(q-1, n)) = 0$, for $p < m$ and $n \gg 0$, follow by standard arguments using either spectral sequences, or by working with a resolution by sums of line bundles for I^{n+1} .

for the first cohomology term, and using the filtration on $\Omega(q-1, n)$ by noting just as above that it can be identified with $\ker(\Omega_{X_{\bar{k}}}^q \otimes I^n/I^{n+1} \rightarrow \Omega_Y^q \otimes I^n/I^{n+1})$.

We have the following analog of Theorem 3.2 in [14].

Proposition 2. *With notation as above, we have for $n \gg 0$,*

$$H^p(Y, \Omega_{(Y_n, Y)}^q) = 0 \text{ for } p < m, \text{ and } p + q < 2m.$$

Proof. We have an exact sequence

$$0 \rightarrow \Omega_{(X_{\bar{k}}, Y_n)}^q \rightarrow \Omega_{(X_{\bar{k}}, Y)}^q \rightarrow i_{n*}\Omega_{(Y_n, Y)}^q \rightarrow 0.$$

On taking cohomology, we have

$$\rightarrow H^p(X_{\bar{k}}, \Omega_{(X_{\bar{k}}, Y)}^q) \rightarrow H^p(Y, \Omega_{(Y_n, Y)}^q) \rightarrow H^{p+1}(X_{\bar{k}}, \Omega_{(X_{\bar{k}}, Y_n)}^q) \rightarrow$$

The first term vanishes by Nori's connectivity theorem, and the last term, for $n \gg 0$ by Proposition 1 in the required range. \square

Corollary 1. $H_{\text{cont}}^p(Y, (\Omega_{Y_n}^q)) \rightarrow H^p(Y, \Omega_Y^q)$ is an isomorphism for $p < m-1$, and $p+q < 2m$, and an injection for $p = m-1$, and $p+q < 2m$.

Proof. We have an exact sequence (see [8])

$$0 \rightarrow R^1 \lim_{\leftarrow n} H^{p-1}(Y_n, \Omega_{Y_n}^q) \rightarrow H_{\text{cont}}^p(Y, (\Omega_{Y_n}^q)) \rightarrow \lim_{\leftarrow n} H^p(Y_n, \Omega_{Y_n}^q) \rightarrow 0.$$

It follows from Proposition 1 above that, for $a \leq m$ and $a+b \leq 2m$, the inverse system $\{H^a(Y, \Omega_{Y_n}^b)\}$ satisfies the Mittag-Leffler condition (in particular, all the transition maps are isomorphisms for $n \gg 0$, and each term is isomorphic to $H^a(X_{\bar{k}}, \Omega_{X_{\bar{k}}}^b)$), hence the first term in the exact sequence vanishes. By Proposition 2, for $n \gg 0$, $H^p(Y_n, \Omega_{Y_n}^q) \rightarrow H^p(Y, \Omega_Y^q)$ is an isomorphism for $p < m-1$, and $p+q < 2m$, and an injection for $p = m-1$, and $p+q < 2m$. Thus we have the desired statement. \square

Finally, we have the following analogue of Theorem 3.10 in [14]. The proof is exactly as in [14] and we leave its verification to the reader. In the following, for any scheme V , we let $\Omega_{V/\mathbb{Q}}^i$ denote the sheaf of absolute Kähler differential q -forms.

Theorem 8. *With notation as above, and $q \geq 1$, the natural restriction map*

$$H_{\text{cont}}^p(Y, (\Omega_{Y_n/\mathbb{Q}}^q / d\Omega_{Y_n/\mathbb{Q}}^{q-1})) \rightarrow H^p(Y, \Omega_{Y/\mathbb{Q}}^q / d\Omega_{Y/\mathbb{Q}}^{q-1})$$

is an isomorphism for $p < m-1$, and $p+q < 2m$, and an injection for $p = m-1$, and $p+q < 2m$.

3. Proof of the main theorem

Theorem 9 (*An infinitesimal Noether–Lefschetz theorem for K-cohomology groups*). *With notation as above, the natural map*

$$H_{\text{cont}}^p(Y, (\mathcal{K}_{q, Y_n})) \rightarrow H^p(Y, \mathcal{K}_{q, Y})$$

is an isomorphism for $p < m$ and $p+q < 2m$, and an injection for $p = m$ and $p+q \leq 2m$.

Proof. The proof is now exactly as in [14]. We sketch it for the sake of completeness. Consider the restriction map $\mathcal{K}_{q,Y_n} \rightarrow \mathcal{K}_{q,Y}$. This is surjective (see [13], see Lemma 5.9). Let $\mathcal{K}_{q,(Y_n,Y)}$ denote its kernel. We have an exact sequence of pro-sheaves

$$0 \rightarrow (\mathcal{K}_{q,(Y_n,Y)}) \rightarrow (\mathcal{K}_{(q,Y_n)}) \rightarrow \mathcal{K}_{q,Y} \rightarrow 0.$$

To prove the theorem, we need to prove that

$$H_{\text{cont}}^p(Y, (\mathcal{K}_{q,(Y_n,Y)})) = 0 \text{ for } p \leq m \text{ and } p+q \leq 2m.$$

For any scheme V , let $\mathcal{HC}_{i,V}^{/\mathbb{Q}}$ denote the i -th *cyclic homology* sheaf relative to \mathbb{Q} , and let $\mathcal{HC}_{i,(Y_n,Y)}^{/\mathbb{Q}}$ be the kernel of the natural map $\mathcal{HC}_{i,Y_n}^{/\mathbb{Q}} \rightarrow \mathcal{HC}_{i,Y}^{/\mathbb{Q}}$ (the fact that this map is a surjection follows by arguing exactly as in the proof of Lemma 5.9 in [13]). By Corollary 2.9 in [14] (this follows from results in [3,4] in the case of algebras, and their extension to schemes in [17] – see §2.3 in [14] for details)

$$H_{\text{cont}}^a(Y, (\mathcal{K}_{b,(Y_n,Y)})) \cong H_{\text{cont}}^a(Y, (\mathcal{HC}_{b-1,(Y_n,Y)}^{/\mathbb{Q}})) \text{ for all } a, b.$$

The term on the right hand side, can be computed using Corollary 2.6 in [14], which is the sheaffification of a result in [1], and states that one has an isomorphism of graded pro-sheaves

$$(\mathcal{HC}_{i,Y_n}^{/\mathbb{Q}}) \cong (\Omega_{Y_n/\mathbb{Q}}^i / d\Omega_{Y_n/\mathbb{Q}}^{i-1}) \oplus \bigoplus_{j \geq 1} (\mathcal{H}_{\text{dR}}^{i-2j}(Y_n/\mathbb{Q})).$$

Finally, by a result of Ogus (see [11]),² we have $\mathcal{H}_{\text{dR}}^c(Y_n/F) \cong \mathcal{H}_{\text{dR}}^c(Y/F)$, for all n and any field F of characteristic 0. Take $F = \mathbb{Q}$; then the above result together with [Theorem 8](#) gives us the desired result. \square

Remark 5. [Theorem 7](#) has the following generalization to complete intersections (the context for Nori's theorem and his conjecture).

Let X be a smooth, projective variety of dimension $m+c$, and let $\mathcal{O}_X(a_1), \dots, \mathcal{O}_X(a_c)$ be sufficiently ample line bundles. Let $S := \prod_{i=1}^c \mathbb{P}(\text{H}^0(X, \mathcal{O}_X(a_i)))$, $A = X \times S$, and $B := \{(x, f_1, \dots, f_c) \in A \mid f_i(x) = 0, i = 1, \dots, c\}$. Let $k = k(S)$, and $Y \subset X_{\bar{k}}$ be the “geometric generic” fiber as in the introduction. Then one has

$H_{\text{cont}}^p(\mathfrak{X}_{\bar{k}}, (\mathcal{K}_{q,Y_n})) \rightarrow H^p(Y, \mathcal{K}_{q,Y})$ is an isomorphism for $p < m$ and $p+q < 2m$, and an injection for $p = m$ and $p+q \leq 2m$. In particular, $\text{CH}_{\text{cont}}^p(\mathfrak{X}_{\bar{k}}) \rightarrow \text{CH}^p(Y)$ is an isomorphism for $p < m$, and a monomorphism for $p = m$.

As suggested by the referee, we will briefly sketch the changes required to adapt the proof of [Theorem 7](#) to this case. The only place where we need some extra work is in showing that [Proposition 1](#) holds when Y is a complete intersection – this follows from [Remark 4](#). The proof now proceeds in exactly the same manner, and with the same proofs.

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² The result is stated only when $F = \mathbb{C}$, but as observed in [14], the proof goes through for any characteristic zero field.

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