ON THE GEOMETRY OF GENERALIZED QUADRICS

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Abstract. Let $\{f_0, \cdots, f_n; g_0, \cdots, g_n\}$ be a sequence of homogeneous polynomials in $2n+2$ variables with no common zeroes in $\mathbb{P}^{2n+1}$ and suppose that the degrees of the polynomials are such that $Q = \sum_{i=0}^{n} f_i g_i$ is a homogeneous polynomial. We shall refer to the hypersurface X defined by Q as a *generalized quadric*. In this note, we prove that generalized quadrics in $\mathbb{P}^{2n+1}_{\mathbb{C}}$ for $n \geq 1$ are reduced.

1. Introduction

Let $\{f_0, \cdots, f_n; g_0, \cdots, g_n\}$ be a sequence of homogeneous polynomials in $2n+2$ variables with no common zeroes in $\mathbb{P}^{2n+1}$ and suppose that the degrees of the polynomials are such that $Q = \sum_{i=0}^{n} f_i g_i$ is a homogeneous polynomial. We shall refer to the hypersurface X defined by Q as a *generalized quadric*. In this note, we prove that generalized quadrics in $\mathbb{P}^{2n+1}_{\mathbb{C}}$ for $n \geq 1$ are reduced.

In characteristic $p > 0$, it is easy to construct generalized quadrics which are non-reduced. By exploiting this fact, low rank vector bundles were constructed on $\mathbb{P}^4$ and $\mathbb{P}^5$ in [3]. Furthermore, in characteristic 0, reducible generalized quadrics exist in $\mathbb{P}^3$; for instance, the hypersurface defined by $X^2Y^2 - Z^2U^2 = 0$, where X, Y, Z, U are the coordinates of $\mathbb{P}^3$, is such a generalized quadric. We do not know any examples of reducible generalized quadrics in higher dimensional projective spaces. However, the question of non-reducedness is settled by our main theorem.

In general, questions regarding irreducibility or reducedness of schemes are difficult. Thus it was surprising to us that reducedness could be proved for such a general class of hypersurfaces. It is conceivable that a purely algebraic proof of this statement can be found, but we were unable to do this. Our proof uses intersection theory and Chern classes over (possibly non-reduced) schemes. The impetus for the argument came from the article of Gruson et. al [1].

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2. Lemmas

We work over the field of complex numbers $\mathbb{C}$. All schemes that we consider will be of finite type over $\mathbb{C}$.

Let X be any scheme and E be any vector bundle on X. Following [2], we may define the Chern-Hodge classes $\mathrm{c}_i(E) \in \mathrm{H}^i(X, \Omega^i_X)$ and the total Chern class $\mathrm{c}(E) = \sum \mathrm{c}_i(E)$. If $\mathcal{F}$ is a coherent sheaf on X which has a finite resolution by vector bundles

$$0 \to \mathrm{P}^\bullet_X \to \mathcal{F} \to 0\,,$$

then we may define

$$\mathrm{c}(\mathcal{F}) = \mathrm{c}(\mathrm{P}^\bullet_X) := \Pi_k\, \mathrm{c}(\mathrm{P}^k_X)^{(-1)^k} \in \oplus\, \mathrm{H}^i(X, \Omega^i_X).$$

For any morphism $f : Y \to X$, we can define $\mathrm{c}^Y(\mathcal{F}) \in \mathrm{H}^\bullet(Y, \Omega^\bullet_Y)$ as $\mathrm{c}(f^*\,\mathrm{P}^\bullet_X)$. In general, this is not $\mathrm{c}(f^*\mathcal{F})$, since $f^*\mathcal{F}$ may not have a finite resolution by vector bundles on Y. However the two coincide if

$$0 \to f^*\,\mathrm{P}^\bullet_X \to f^*\mathcal{F} \to 0$$

remains exact.

Let $f : Y \to X$ be a morphism, and

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be a short exact sequence of sheaves on X. Assume further that each of the sheaves in the sequence has a finite locally free resolution. Then the Whitney sum formula yields $\mathrm{c}^Y(\mathcal{F}) = \mathrm{c}^Y(\mathcal{F}')\,\mathrm{c}^Y(\mathcal{F}'')$.

The following lemma, which is the key lemma, is essentially due to Gruson et.al [1].

Lemma 1. *Let $X \subset \mathbb{P}^n$ be an irreducible hypersurface which is not reduced. Consider the restriction maps*

$$\mathrm{H}^i(\Omega^i_{\mathbb{P}^n}) \xrightarrow{\alpha} \mathrm{H}^i(\Omega^i_{X_{\mathrm{red}}})$$

and

$$\mathrm{H}^i(\Omega^i_X) \xrightarrow{\beta} \mathrm{H}^i(\Omega^i_{X_{\mathrm{red}}}).$$

Then $\mathrm{Im}\,\beta = \mathrm{Im}\,\alpha$ for $1 \le i < n-1$.

Proof. Since α factors through $\mathrm{H}^i(\Omega^i_X)$, we only need to show that $\mathrm{Im}\beta \subset \mathrm{Im}\alpha$. Since X is irreducible, we may assume that X is defined by a homogeneous polynomial $f^m, m > 1$ with f irreducible and so X_{red} is given by the vanishing of f. We consider the exact sequence

$$\mathcal{O}_X(-\deg(f^m)) \xrightarrow{d(f^m)} \Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_X \to \Omega^1_X \to 0$$

Restricting it to X_{red}, we get

$$\Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\mathrm{red}}} \cong \Omega^1_X \otimes \mathcal{O}_{X_{\mathrm{red}}} \tag{1}$$

This implies similar isomorphisms,

$$\Omega^i_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\rm red}} \cong \Omega^i_X \otimes \mathcal{O}_{X_{\rm red}},$$

for all i.

Since α factors through $\mathrm{H}^i(\Omega^i_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\rm red}})$ and similarly β factors through $\mathrm{H}^i(\Omega^i_X \otimes \mathcal{O}_{X_{\rm red}})$, it suffices to prove that the map

$$\mathrm{H}^i(\Omega^i_{\mathbb{P}^n}) \xrightarrow{\delta} \mathrm{H}^i(\Omega^i_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\rm red}})$$

is onto by the isomorphism (1) above. We have an exact sequence,

$$0 \to \Omega^i_{\mathbb{P}^n}(-d) \to \Omega^i_{\mathbb{P}^n} \to \Omega^i_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\rm red}} \to 0,$$

where $d = \deg f$. Taking cohomologies and noting that (see [5] page 8, for instance) $\mathrm{H}^j(\Omega^i_{\mathbb{P}^n}(-d)) = 0$ for $j = i, i+1$, since $1 \le i < n-1$, we see that δ is an isomorphism.

□

Lemma 2. *Let $M \subset \mathbb{P}^n$ be a closed subscheme of dimension r. Then the natural map,*

$$\gamma : \mathrm{H}^i(\Omega^i_{\mathbb{P}^n}) \to \mathrm{H}^i(\Omega^i_M)$$

is injective for $0 \le i \le r$.

Proof. If $h \in \mathrm{H}^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n})$ is the class of the hyperplane section, then $\mathrm{H}^i(\mathbb{P}^n, \Omega^i_{\mathbb{P}^n})$ is a one dimensional vector space generated by h^i. Thus, it suffices to show that its image in $\mathrm{H}^i(M, \Omega^i_M)$ is non-zero. If it is zero for some $i < r$, then $h^r = h^i h^{r-i} = 0 \in \mathrm{H}^r(M, \Omega^r_M)$ and so $\deg M = 0$. This is a contradiction since the degree of a variety is always positive. □

Lemma 3. *Let $X \subset \mathbb{P}^n$ be an irreducible hypersurface which is not reduced. Let $\mathcal{F}$ be a coherent sheaf on X with a resolution $0 \to \mathrm{P}^\bullet_X \to \mathcal{F} \to 0$ by vector bundles on X such that $0 \to \mathrm{P}^\bullet_X \otimes \mathcal{O}_M \to 0$ is exact where $M \subset X_{\rm red}$ and $\dim M = r$. Then $0 = \mathrm{c}_i^{X_{\rm red}}(\mathcal{F}) \in \mathrm{H}^i(X, \Omega^i_{X_{\rm red}})$ for $1 \le i \le \min\{r, n-2\}$.*

Proof. Since $0 \to \mathrm{P}^\bullet_X \otimes \mathcal{O}_M \to 0$ is exact, we have $\mathrm{c}^M(\mathcal{F}) = 1$.From Lemma 1 above, it follows that $\forall\ 1 \le i \le \min\{r, n-2\}$, there exist classes $t_i \in \mathrm{H}^i(\Omega^i_{\mathbb{P}^n})$ such that

$$\beta(\mathrm{c}_i(\mathcal{F})) = \alpha(t_i).$$

Let $\theta : \mathrm{H}^i(\Omega^i_{X_{\rm red}}) \to \mathrm{H}^i(\Omega^i_M)$ be the natural map. Then $\theta\beta(\mathrm{c}_i(\mathcal{F})) = \mathrm{c}_i^M(\mathcal{F}) = 0$ for $i > 0$. Thus $\theta\alpha(t_i) = 0$. But $\theta\alpha = \gamma$ and by Lemma 2, we get that $t_i = 0$ for $1 \le i \le \min\{r, n-2\}$ and thus

$$\mathrm{c}_i^{X_{\rm red}}(\mathcal{F}) = \beta\, \mathrm{c}_i(\mathcal{F}) = 0$$

for $1 \le i \le \min\{r, n-2\}$. □

3. Generalized Quadrics

In this section, we apply the results of the previous section to show that generalized quadrics in $\mathbb{P}^{2n+1}$ for $n \geq 1$ are reduced.

Let $Q \subset \mathbb{P}^{2n+1}$ denote the generalized quadric given by the equation $\sum_{i=0}^{n} f_i g_i = 0$. Let

$$Z := Q \cap (f_1 = \cdots = f_n = 0)$$
$$L_1 := (f_0 = \cdots = f_n = 0)$$
$$L_2 := (g_0 = f_1 = \cdots = f_n = 0).$$

Note that L_1 and L_2 are also subschemes of Q. Then $Z = L_1 \cup L_2$ and we have an exact sequence

$$0 \to \mathcal{O}_{L_2}(-\deg f_0) \to \mathcal{O}_Z \to \mathcal{O}_{L_1} \to 0.$$

Furthermore, Z is a complete intersection of n ample divisors on Q, L_i for $i = 1, 2$ are local complete intersection subschemes in Q of codimension (and dimension) n.

Theorem 1. *The generalized quadric Q is reduced.*

Proof. If Q is not reduced, let X be an irreducible component of Q which is not reduced and let X_{red} denote the subscheme X with the reduced structure. Thus $\sum f_i g_i = f^r f'$ with f an irreducible polynomial, $r > 1$ where $f^r = 0$ defines X and $f = 0$ defines X_{red}.

Let $Z' = Z \cap X$, $L'_i = L_i \cap X$. It is easy to see that Z' is a complete intersection in X by f_i, $i > 0$. Let $a_i = \operatorname{degree} f_i$. We consider the Koszul resolution of $\mathcal{O}_{Z'}$ on X given by the f_i's:

$$0 \to \mathcal{O}_X(-\sum_i a_i) \to \cdots \to \oplus_i \mathcal{O}_X(-a_i) \to \mathcal{O}_X \to \mathcal{O}_{Z'} \to 0.$$

By an easy computation, it follows that

$$\mathrm{c}_n(\mathcal{O}_{Z'}) = a h^n \in \mathrm{H}^n(\Omega^n_X)$$

where $a = (-1)^{n-1}(n-1)!\,(\Pi_i a_i) \neq 0$.

On the other hand, since L'_i are local complete intersections in X, there exist finite resolutions by vector bundles over X for the sheaves $\mathcal{O}_{L'_i}$:

$$0 \to \mathrm{P}^{\bullet}_i \to \mathcal{O}_{L'_i} \to 0.$$

We have an exact sequence,

$$0 \to \mathcal{O}_{L'_2}(-d) \to \mathcal{O}_{Z'} \to \mathcal{O}_{L'_1} \to 0$$

where $d = \deg f_0$. By the Whitney sum formula this implies that

$$\mathrm{c}(\mathcal{O}_{Z'}) = \mathrm{c}(\mathcal{O}_{L'_1})\,\mathrm{c}(\mathcal{O}_{L'_2}(-d)) \in \mathrm{H}^{\bullet}(\Omega^{\bullet}_X).$$

Let M_1 be the subscheme defined by the vanishing of $g_0, \ldots, g_n$ in X_{red}. Then $\dim M_1 = n$ and since $L_1 \cap M_1 = \emptyset$, we get, $0 \to \mathrm{P}_1^\bullet \otimes \mathcal{O}_{M_1} \to 0$ is exact. Since $\dim M_1 = n = \min\{n, 2n+1-2\}$, by Lemma 3, we see that $\mathrm{c}^{X_{\text{red}}}(\mathcal{O}_{L_1'}) = 1 + x$, where $x \in \oplus_{i>n} \mathrm{H}^i(\Omega^i_{X_{\text{red}}})$. A similar argument with L_2 and M_2 (which is defined by the vanishing of $f_0, g_1, \ldots, g_n$ on X_{red}) gives $\mathrm{c}^{X_{\text{red}}}(\mathcal{O}_{L_2'}(-d)) = 1 + y$ where $y \in \oplus_{i>n} \mathrm{H}^i(\Omega^i_{X_{\text{red}}})$. By applying the Whitney sum formula, we have $\mathrm{c}^{X_{\text{red}}}(\mathcal{O}_{Z'}) = 1 + z$ with $z \in \oplus_{i>n} \mathrm{H}^i(\Omega^i_{X_{\text{red}}})$. In particular, we see that $c_n^{X_{\text{red}}}(\mathcal{O}_{Z'}) = 0$. But, we have seen that this is the image of ah^n for $a \neq 0$, h the class of hyperplane section. By Lemma 2, this is a contradiction. □

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