ON THE GEOMETRY OF GENERALISED QUADRICS

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Abstract. Let $\{f_0, \cdots, f_n; g_0, \cdots, g_n\}$ be a sequence of homogeneous polynomials in $2n+2$ variables with no common zeroes in $\mathbb{P}^{2n+1}$ and suppose that the degrees of the polynomials are such that $Q = \sum_{i=0}^{n} f_i g_i$ is a homogeneous polynomial. We shall refer to the hypersurface X defined by Q as a *generalised quadric*. In this note, we prove that generalised quadrics in $\mathbb{P}^{2n+1}_{\mathbb{C}}$ for $n \geq 1$ are reduced.

1. Introduction

Let $\{f_0, \cdots, f_n; g_0, \cdots, g_n\}$ be a sequence of homogeneous polynomials in $2n+2$ variables with no common zeroes in $\mathbb{P}^{2n+1}$ and suppose that the degrees of the polynomials are such that $Q = \sum_{i=0}^{n} f_i g_i$ is a homogeneous polynomial. We shall refer to the hypersurface X defined by Q as a *generalised quadric*. In this note, we prove that generalised quadrics in $\mathbb{P}^{2n+1}_{\mathbb{C}}$ for $n \geq 1$ are reduced.

In characteristic $p > 0$, it is easy to construct generalised quadrics which are non-reduced. By exploiting this fact, low rank vector bundles were constructed on $\mathbb{P}^4$ and $\mathbb{P}^5$ in [4]. Furthermore, in characteristic 0, reducible generalised quadrics exist in $\mathbb{P}^3$; for instance, the hypersurface defined by $X^2Y^2 - Z^2U^2 = 0$, where X, Y, Z, U are the coordinates of $\mathbb{P}^3$, is such a generalised quadric. We do not know any examples of reducible generalised quadrics in higher dimensional projective spaces. However, the question of non-reducedness is settled by our main theorem.

In general, questions regarding irreducibility or reducedness of schemes are difficult. Thus it was surprising to us that reducedness could be proved for such a general class of hypersurfaces. It is conceivable that a purely algebraic proof of this statement can be found, but we were unable to do this. Our proof uses intersection theory and Chern classes

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over (possibly non-reduced) schemes. The impetus for the argument came from the article of Ellingsrud et al. [2]

2. Atiyah Class and Chern classes of vector bundles over schemes

We work over the field of complex numbers $\mathbb{C}$. All schemes that we consider will be of finite type over $\mathbb{C}$.

Let X be any scheme and E be any vector bundle on X. We recall that the *Atiyah class* $\mathrm{at}(E)$ (see [1]) of the vector bundle E is the natural extension class

$$0 \to \Omega^1_X \otimes E \to \mathcal{P}(E) \to E \to 0$$

where $\mathcal{P}(E)$ is the *principal parts bundle* of E. Thus $\mathrm{at}(E)$ is an element of the cohomology group $\mathrm{H}^1(X, \Omega^1_X \otimes \mathcal{E}nd(E))$. Starting with this class, one can define *Chern-Hodge classes* $\mathrm{c}_i(E) \in \mathrm{H}^i(X, \Omega^i_X)$ as follows (see [3] or for a simpler exposition see [5]).

Consider the composition

$$(\Omega^1_X \otimes \mathcal{E}nd(E))^{\otimes m} \to (\Omega^1_X)^{\otimes m} \otimes \mathcal{E}nd(E^{\otimes m}) \to \Omega^m_X \otimes \mathcal{E}nd(\overset{\mathrm{m}}{\wedge} E) \to \Omega^m_X$$

where the first map and the map $\Omega^{1\otimes m}_X \to \Omega^m_X$ are the obvious ones, the last map is induced by the trace map $\mathcal{E}nd(\overset{\mathrm{m}}{\wedge} E) \to \mathcal{O}_X$ and the map $\mathcal{E}nd(E^{\otimes m}) \to \mathcal{E}nd(\overset{\mathrm{m}}{\wedge} E)$ in the middle is defined as $f \mapsto \pi_E \circ f \circ j$ where $\pi_E : E^{\otimes m} \to \overset{\mathrm{m}}{\wedge} E$ is the natural projection and $j : \overset{\mathrm{m}}{\wedge} E \to E^{\otimes m}$ is the map $\frac{1}{m!}(\pi_{E^\vee})^\vee$.

We then define the Chern-Hodge classes from the composite map below:

$$\begin{array}{ccccc} \mathrm{H}^1(X, \Omega^1_X \otimes \mathcal{E}nd(E))^{\otimes m} & \to & \mathrm{H}^m(X, (\Omega^1_X \otimes \mathcal{E}nd(E))^{\otimes m}) & \to & \mathrm{H}^m(X, \Omega^m_X) \\ \mathrm{at}(E)^{\otimes m} & \mapsto & \mathrm{at}(E) \cup \cdots \cup \mathrm{at}(E) & \mapsto & \mathrm{c}_m(E) \end{array}$$

By convention, $c_0(E) = 1 \in \mathrm{H}^0(X, \mathcal{O}_X)$. Furthermore, we let $\mathrm{c}(E) = \sum \mathrm{c}_i(E)$ which is an invertible element in the graded commutative ring $\oplus_i \mathrm{H}^i(X, \Omega^i_X)$.

Now let X be any (finite type) scheme and let $\mathcal{F}$ be a coherent sheaf on X which has a finite resolution by vector bundles

$$0 \to \mathrm{P}^\bullet_X \to \mathcal{F} \to 0$$

Definition 1. $\mathrm{c}(\mathcal{F}) = \mathrm{c}(\mathrm{P}^\bullet_X) := \Pi_k\, \mathrm{c}(\mathrm{P}^k_X)^{(-1)^k} \in \oplus\, \mathrm{H}^i(X, \Omega^i_X)$.

We recall some basic properties of the Chern-Hodge classes. Let $\mathcal{P}(X)$ [1] be the set of all sheaves on X which have a finite resolution by vector bundles.

Properties:

1. For any sheaf $\mathcal{F} \in \mathcal{P}(X)$, $\mathrm{c}(\mathcal{F})$ is independent of the resolution.
2. For any short exact sequence of sheaves in $\mathcal{P}(X)$
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
$\mathrm{c}(\mathcal{F}) = \mathrm{c}(\mathcal{F}')\,\mathrm{c}(\mathcal{F}'')$.
3. For any morphism $f : Y \to X$, there is a natural ring homomorphism $f^* : \oplus_i \mathrm{H}^i(X, \Omega^i_X) \to \oplus_i \mathrm{H}^i(Y, \Omega^i_Y)$ under which if E is a bundle on X, then $f^*\,\mathrm{c}(E) = \mathrm{c}(f^*E)$.
4. For any bundle E and a line bundle $\mathcal{L}$, we have
$$\mathrm{c}_r(E \otimes \mathcal{L}) = \sum_{i=0}^{r} \mathrm{c}_i(E)\,\mathrm{c}_1(\mathcal{L}^{r-i})$$
5. If $\mathcal{F} \in \mathcal{P}(X)$ and
$$0 \to \mathrm{P}^\bullet_X \to \mathcal{F} \to 0$$
is a finite resolution by vector bundles and if $f : Y \to X$ is any morphism, we can define $\mathrm{c}^Y(\mathcal{F}) \in \mathrm{H}^\bullet(Y, \Omega^\bullet_Y)$ as $\mathrm{c}(f^*\,\mathrm{P}^\bullet_X)$. In general, this is not $\mathrm{c}(f^*\mathcal{F})$, since this sheaf may not have a finite resolution by vector bundles on Y. These coincide if
$$0 \to f^*\,\mathrm{P}^\bullet_X \to f^*\mathcal{F} \to 0$$
remains exact and thus in this case $\mathrm{c}^Y(\mathcal{F}) = \mathrm{c}(f^*\mathcal{F})$.
6. For any short exact sequence of sheaves in $\mathcal{P}(X)$
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
on X and a morphism $f : Y \to X$, $\mathrm{c}^Y(\mathcal{F}) = \mathrm{c}^Y(\mathcal{F}')\,\mathrm{c}^Y(\mathcal{F}'')$.

The following lemma, which is the key lemma, is essentially due to Ellingsrud et al. [2]

Lemma 1. *Let $X \subset \mathbb{P}^n$ be an irreducible hypersurface which is not reduced. Consider the restriction maps*
$$\mathrm{H}^i(\Omega^i_{\mathbb{P}^n}) \xrightarrow{\alpha} \mathrm{H}^i(\Omega^i_{X_{\mathrm{red}}})$$
and
$$\mathrm{H}^i(\Omega^i_X) \xrightarrow{\beta} \mathrm{H}^i(\Omega^i_{X_{\mathrm{red}}}).$$
Then $\operatorname{Im}\beta = \operatorname{Im}\alpha$ *for* $1 \le i < n-1$.

[1] One could instead work with the more standard setting of perfect complexes.

Proof. Since α factors through $\mathrm{H}^i(\Omega^i_X)$, we only need to show that $\mathrm{Im}\beta \subset \mathrm{Im}\alpha$. Since X is irreducible, we may assume that X is defined by a homogeneous polynomial $f^m, m > 1$ with f irreducible and so X_{red} is given by the vanishing of f. We consider the exact sequence

$$\mathcal{O}_X(-\deg(f^m)) \xrightarrow{d(f^m)} \Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_X \to \Omega^1_X \to 0$$

Restricting it to X_{red}, we get

$$\Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\mathrm{red}}} \cong \Omega^1_X \otimes \mathcal{O}_{X_{\mathrm{red}}} \tag{1}$$

This implies similar isomorphisms,

$$\Omega^i_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\mathrm{red}}} \cong \Omega^i_X \otimes \mathcal{O}_{X_{\mathrm{red}}},$$

for all i.

Since α factors through $\mathrm{H}^i(\Omega^i_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\mathrm{red}}})$ and similarly β factors through $\mathrm{H}^i(\Omega^i_X \otimes \mathcal{O}_{X_{\mathrm{red}}})$, it suffices to prove that the map

$$\mathrm{H}^i(\Omega^i_{\mathbb{P}^n}) \xrightarrow{\delta} \mathrm{H}^i(\Omega^i_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\mathrm{red}}})$$

is onto by the isomorphism (1) above. We have an exact sequence,

$$0 \to \Omega^i_{\mathbb{P}^n}(-d) \to \Omega^i_{\mathbb{P}^n} \to \Omega^i_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\mathrm{red}}} \to 0,$$

where $d = \deg f$. Taking cohomologies and noting that (see [6] page 8, for instance) $\mathrm{H}^j(\Omega^i_{\mathbb{P}^n}(-d)) = 0$ for $j = i, i+1$, since $1 \leq i < n-1$, we see that δ is an isomorphism.

□

Lemma 2. *Let $M \subset \mathbb{P}^n$ be a closed subscheme of dimension r. Then the natural map,*

$$\gamma : \mathrm{H}^i(\Omega^i_{\mathbb{P}^n}) \to \mathrm{H}^i(\Omega^i_M)$$

is injective for $0 \leq i \leq r$.

Proof. If $h \in \mathrm{H}^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n})$ is the class of the hyperplane section, then $\mathrm{H}^i(\mathbb{P}^n, \Omega^i_{\mathbb{P}^n})$ is a one dimensional vector space generated by h^i. Thus, it suffices to show that its image in $\mathrm{H}^i(M, \Omega^i_M)$ is non-zero. If it is zero for some $i < r$, then $h^r = h^i h^{r-i} = 0 \in \mathrm{H}^r(M, \Omega^r_M)$. A proof of the well known fact that $h^r \neq 0$ is sketched in the Appendix. □

Lemma 3. *Let $X \subset \mathbb{P}^n$ be an irreducible hypersurface which is not reduced. Let $\mathcal{F}$ be a coherent sheaf on X with a resolution $0 \to \mathrm{P}^\bullet_X \to \mathcal{F} \to 0$ by vector bundles on X such that $0 \to \mathrm{P}^\bullet_X \otimes \mathcal{O}_M \to 0$ is exact where $M \subset X_{\mathrm{red}}$ and $\dim M = r$. Then $0 = \mathrm{c}_i^{X_{\mathrm{red}}}(\mathcal{F}) \in \mathrm{H}^i(X, \Omega^i_{X_{\mathrm{red}}})$ for $1 \leq i \leq \min\{r, n-2\}$.*

Proof. Since $0 \to \mathrm{P}^{\bullet}_X \otimes \mathcal{O}_M \to 0$ is exact, $\mathrm{c}^M(\mathcal{F}) = 1$ by Property 5. From Lemma 1 above, it follows that $\forall\ 1 \le i \le \min\{r, n-2\}$, there exist classes $t_i \in \mathrm{H}^i(\Omega^i_{\mathbb{P}^n})$ such that

$$\beta(\mathrm{c}_i(\mathcal{F})) = \alpha(t_i).$$

Let $\theta : \mathrm{H}^i(\Omega^i_{X_{\mathrm{red}}}) \to \mathrm{H}^i(\Omega^i_M)$ be the natural map. Then $\theta\beta(\mathrm{c}_i(\mathcal{F})) = \mathrm{c}^M_i(\mathcal{F}) = 0$ for $i > 0$. Thus $\theta\alpha(t_i) = 0$. But $\theta\alpha = \gamma$ and by Lemma 2, we get that $t_i = 0$ for $1 \le i \le \min\{r, n-2\}$ and thus

$$\mathrm{c}^{X_{\mathrm{red}}}_i(\mathcal{F}) = \beta\,\mathrm{c}_i(\mathcal{F}) = 0$$

for $1 \le i \le \min\{r, n-2\}$. □

3. Generalised Quadrics

In this section, we apply the results of the previous section to show that generalised quadrics in $\mathbb{P}^{2n+1}$ for $n \ge 1$ are reduced.

Let $Q \subset \mathbb{P}^{2n+1}$ denote the generalised quadric given by the equation $\sum_{i=0}^n f_i g_i = 0$. Let

$$Z := Q \cap (f_1 = \cdots = f_n = 0)$$
$$L_1 := (f_0 = \cdots = f_n = 0)$$
$$L_2 := (g_0 = f_1 = \cdots = f_n = 0).$$

Note that L_1 and L_2 are also subschemes of Q. Then $Z = L_1 \cup L_2$ and we have an exact sequence

$$0 \to \mathcal{O}_{L_2}(-\deg f_0) \to \mathcal{O}_Z \to \mathcal{O}_{L_1} \to 0.$$

Furthermore, Z is a complete intersection of n ample divisors on Q, L_i for $i = 1, 2$ are local complete intersection subschemes in Q of codimension (and dimension) n.

Theorem 1. *The generalised quadric Q is reduced.*

Proof. If Q is not reduced, let X be an irreducible component of Q which is not reduced and let X_{red} denote the subscheme X with the reduced structure. Thus $\sum f_i g_i = f^r f'$ with f an irreducible polynomial, $r > 1$ where $f^r = 0$ defines X and $f = 0$ defines X_{red}.

Let $Z' = Z \cap X$, $L'_i = L_i \cap X$. It is easy to see that Z' is a complete intersection in X by f_i, $i > 0$. Let $a_i = \mathrm{degree}\, f_i$. We consider the Koszul resolution of $\mathcal{O}_{Z'}$ on X given by the f_i's:

$$0 \to \mathcal{O}_X(-\sum_i a_i) \to \cdots \to \oplus_i \mathcal{O}_X(-a_i) \to \mathcal{O}_X \to \mathcal{O}_{Z'} \to 0.$$

By an easy computation, it follows that

$$\mathrm{c}_n(\mathcal{O}_{Z'}) = a h^n \in \mathrm{H}^n(\Omega^n_X)$$

where $a = (-1)^{n-1}(n-1)!\ (\Pi_i a_i) \neq 0$.

On the other hand, since L'_i are local complete intersections in X, there exist finite resolutions by vector bundles over X for the sheaves $\mathcal{O}_{L'_i}$:

$$0 \to \mathrm{P}_i^\bullet \to \mathcal{O}_{L'_i} \to 0.$$

We have an exact sequence,

$$0 \to \mathcal{O}_{L'_2}(-d) \to \mathcal{O}_{Z'} \to \mathcal{O}_{L'_1} \to 0$$

where $d = \deg f_0$ which gives by Property 2 that

$$\mathrm{c}(\mathcal{O}_{Z'}) = \mathrm{c}(\mathcal{O}_{L'_1})\,\mathrm{c}(\mathcal{O}_{L'_2}(-d)) \in \mathrm{H}^\bullet(\Omega_X^\bullet).$$

Let M_1 be the subscheme defined by the vanishing of $g_0, \dots, g_n$ in X_{red}. Then $\dim M_1 = n$ and since $L_1 \cap M_1 = \emptyset$, we get, $0 \to \mathrm{P}_1^\bullet \otimes \mathcal{O}_{M_1} \to 0$ is exact. Since $\dim M_1 = n = \min\{n, 2n+1-2\}$, by Lemma 3, we see that $\mathrm{c}^{X_{\mathrm{red}}}(\mathcal{O}_{L'_1}) = 1 + x$, where $x \in \oplus_{i>n} \mathrm{H}^i(\Omega^i_{X_{\mathrm{red}}})$. A similar argument with L_2 and M_2 (which is defined by the vanishing of $f_0, g_1, \dots, g_n$ on X_{red}) gives $\mathrm{c}^{X_{\mathrm{red}}}(\mathcal{O}_{L'_2}(-d)) = 1 + y$ where $y \in \oplus_{i>n} \mathrm{H}^i(\Omega^i_{X_{\mathrm{red}}})$. Thus by Property 6, $\mathrm{c}^{X_{\mathrm{red}}}(\mathcal{O}_{Z'}) = 1 + z$ with $z \in \oplus_{i>n} \mathrm{H}^i(\Omega^i_{X_{\mathrm{red}}})$. In particular, we see that $c_n^{X_{\mathrm{red}}}(\mathcal{O}_{Z'}) = 0$. But, we have seen that this is the image of ah^n for $a \neq 0$, h the class of hyperplane section. By Lemma 2, this is a contradiction. $\square$

4. Appendix

As before, we shall work over $\mathbb{C}$. The purpose of this appendix is to prove the following theorem which is folklore, but we give a proof for completeness.

Theorem 2. *Let X be a projective scheme over $\mathbb{C}$ of dimension $r \geq 1$, $h \in \mathrm{H}^1(X, \Omega^1_X)$ the class of a hyperplane section. Then h^r in $\mathrm{H}^r(X, \Omega^r_X)$ is not zero.*

Let $h \in \mathrm{H}^1(\mathbb{P}^r, \Omega^1_{\mathbb{P}^r})$ be the class of a hyperplane. We will assume the well known facts that $h^i \in \mathrm{H}^i(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r})$ generates this one dimensional vector space (in particular $h^i \neq 0$) and $\mathrm{c}_1(\mathcal{O}_{\mathbb{P}^r}(1))$ is a non-zero multiple of h.

Theorem 3. *Let H be a hyperplane section of $\mathbb{P}^r$ with $r \geq 1$. Then we have a canonical isomorphism $\alpha : \mathrm{H}^{r-1}(H, \Omega^{r-1}_H) \to \mathrm{H}^r(\mathbb{P}^r, \Omega^r_{\mathbb{P}^r})$.*

Proof. We have a canonical exact sequence,

$$0 \to \Omega^r_{\mathbb{P}^r} \to \Omega^r_{\mathbb{P}^r}(H) \to \Omega^{r-1}_H \to 0.$$

This gives, by taking cohomologies an isomorphism

$$\alpha : \mathrm{H}^{r-1}(H, \Omega^{r-1}_H) \to \mathrm{H}^r(\mathbb{P}^r, \Omega^r_{\mathbb{P}^r}),$$

using the fact (see [6]) that $\mathrm{H}^i(\mathbb{P}^r, \Omega^r_{\mathbb{P}^r}(H)) = 0$ for $i = r-1, r$. $\square$

Lemma 4. *Let C be a non-singular projective curve and let L be an ample line bundle. Then $l = \mathrm{c}_1(L) \in \mathrm{H}^1(C, \Omega^1_C)$ is not zero.*

Proof. It is clear that we may replace l by nl for any $n > 0$ and thus we may assume that L is very ample. This gives, by taking two general sections of L, a morphism $f : C \to \mathbb{P}^1$ with $f^*(\mathcal{O}_{\mathbb{P}^1}(1)) = L$. Since

$$l = \mathrm{c}_1(L) = \mathrm{c}_1(f^*(\mathcal{O}_{\mathbb{P}^1}(1))) = f^*(\mathrm{c}_1(\mathcal{O}_{\mathbb{P}^1}(1))),$$

it suffices to prove that $\mathrm{c}_1(\mathcal{O}_{\mathbb{P}^1}(1)) \neq 0$ which we have assumed and that $f^* : \mathrm{H}^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) \to \mathrm{H}^1(C, \Omega^1_C)$ is injective. The second statement is obvious, since the natural map $\Omega^1_{\mathbb{P}^1} \to f_*\Omega^1_C$ splits. $\square$

Proof of Theorem 2. Let $Y \subset X$ be a reduced irreducible closed subvariety of dimension r. We have a natural map $\mathrm{H}^r(X, \Omega^r_X) \to \mathrm{H}^r(Y, \Omega^r_Y)$. Thus it suffices to prove the theorem for Y, since h^r goes to h^r. Thus we may assume that X is integral. Similarly, we may replace X by its normalization and thus assume that X is normal. Proof is by induction on r where the case $r = 1$ is treated in lemma 4.

For the induction step we proceed as follows. If h is the class of the ample line bundle H, we may clearly replace H by nH, $n > 0$. Thus we may assume that $\mathrm{H}^i(X, \Omega^r_X(H)) = 0$ for $i = r-1, r$, since $r \geq 2$. Further, by Bertini theorems, we have a section $Y \in |H|$ which is integral and normal and the multiplication map $Y : \Omega^r_X \to \Omega^r_X(H)$ is injective. Let $\mathcal{E}$ be the cokernel of this map. We may also assume that we have a finite map $f : X \to \mathbb{P}^r$ such that $f^*(\mathcal{O}_{\mathbb{P}^r}(1)) = H$ and a section $s \in \mathrm{H}^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$ such that f^*s corresponds to Y. Let us denote the hyperplane $s = 0$ by L. By our assumption, we see that the map $\mathrm{H}^{r-1}(Y, \mathcal{E}) \to \mathrm{H}^r(X, \Omega^r_X)$ is an isomorphism. We also see that on the smooth points of Y, $\mathcal{E} \cong \Omega^{r-1}_Y$. This says that the double dual of $\mathcal{E}$ and Ω^{r-1}_Y are isomorphic. We denote the double dual by $\mathcal{F}$. Thus we have maps $\mathcal{E} \to \mathcal{F}$ and $\Omega^{r-1}_Y \to \mathcal{F}$ which are isomorphisms on the open subset of smooth points. Since the codimension of the singular locus is at least 2, we see that

$$\mathrm{H}^{r-1}(Y, \mathcal{E}) \cong \mathrm{H}^{r-1}(Y, \mathcal{F}) \cong \mathrm{H}^{r-1}(Y, \Omega^{r-1}_Y).$$

Using f, we have a commutative diagram,

$$\begin{array}{ccc} \mathrm{H}^{r-1}(L, \Omega^{r-1}_L) & \xrightarrow{\cong} & \mathrm{H}^r(\mathbb{P}^r, \Omega^r_{\mathbb{P}^r}) \\ \downarrow f^* & & \downarrow f^* \\ \mathrm{H}^{r-1}(Y, \mathcal{E}) & \xrightarrow{\cong} & \mathrm{H}^r(X, \Omega^r_X) \end{array}$$

We have the natural map $\mathrm{H}^{r-1}(L, \Omega^{r-1}_L) \xrightarrow{f^*} \mathrm{H}^{r-1}(Y, \Omega^{r-1}_Y)$ and the class of h^{r-1} goes to a non-zero element by induction. But the latter group is isomorphic to $\mathrm{H}^{r-1}(Y, \mathcal{E})$ and thus h^{r-1} goes to a non-zero element in this group and then by the above isomorphism, its image in $\mathrm{H}^r(X, \Omega_X^r)$ is non-zero. Now, following h^{r-1} via the other branch of the commutative diagram, we see that $h^r \neq 0$ in $\mathrm{H}^r(X, \Omega_X^r)$ by theorem 3.

$\square$

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